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OBSERVATIONS ON THE HOMOGENOURS TERNARY QUADRATIC DIOPHANTINE EQUATION $39(X^2+y^2)-72Xy=246z^2$

Dr. A. Vijayasankar[1], Sharadha Kumar[2], & Dr. M. A. Gopalan[3]

[1]*Assistant Professor, Department of Mathematics, National College, Affiliated to Bharathidasan University, Trichy, Tamil Nadu, India*

[2]*Research Scholar, Department of Mathematics, National College, Affiliated to Bharathidasan University ,Trichy, Tamil Nadu, India*

[3]*Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy, Tamil Nadu, India*

ABSTRACT

The homogeneous ternary quadratic equation given by $39(x^2+y^2)-72xy=246z^2$ *is analysed for its non-zero distinct integer solutions through different methods. Also, formulae for generating sequence of integer solutions based on the given solution are presented.*

KEYWORDS: *Ternary Quadratic, Homogeneous Quadratic, Integer Solutions*

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INTRODUCTION

Ternary quadratic equations are rich in variety [1- 4, 17-19]. For an extensive review of sizable literature and various problems, one may refer [5-16]. In this communication, yet another interesting homogeneous ternary quadratic Diophantine equation given by $39(x^2+y^2)-72xy=246z^2$ is analysed for its non-zero distinct integer solutions through different methods. Also, formulas for generating sequence of integer solutions based on the given solution are presented.

Method of Analysis

The homogenous ternary quadratic Diophantine Equation is

$$39(x^2+y^2)-72xy=246z^2 \tag{1}$$

Consider the linear transformations

$$x=u+v,\, y=u-v,\, u\neq v\neq 0 \tag{2}$$

Using (2) in (1) and simplifying, one obtains

$$u^2+25v^2=41z^2 \tag{3}$$

Solving (3) for u_1v and z through different ways and employing (2), different sets of nom-zero distinct integer solutions to (1) are obtained. The different ways of solving (3) and the corresponding integer solutions to (1) are illustrated below.

Way 1

Assume $v = 41a^2 - b^2$ (4)

Write 25 as

$$25 = (\sqrt{41} + 4)(\sqrt{41} - 4) \tag{5}$$

Substituting (4) and (5) in (3) and applying the method of factorization, define

$$\sqrt{41}z + u = (\sqrt{41} + 4)(\sqrt{41}a + b)^2 \tag{6}$$

Equating the rational and irrational parts,

$$u = 4(41a^2 + b^2) + 82ab \tag{7}$$

$$z = 41a^2 + b^2 + 8ab \tag{8}$$

Employing (4) and (7) in (2), one has

$$\left.\begin{aligned} x &= 205a^2 + 3b^2 + 82ab \\ y &= 123a^2 + 5b^2 + 82ab \end{aligned}\right\} \tag{9}$$

Thus, (8) and (9) give the non-zero distinct integer solutions to (1)

Way 2

Introducing the linear transformations

$$z = X + 25T, v = X + 41T, u = 4U \tag{10}$$

in (3), it is written as

$$X^2 = 1025T^2 + U^2 \tag{11}$$

Which is satisfied by

$$T = 2rs, X = 1025r^2 + s^2, U = 1025r^2 - s^2 \tag{12}$$

From (12), (10) and (2), observe that

$$x = 5125r^2 - 3s^2 + 82rs$$
$$y = 3075r^2 - 5s^2 - 82rs$$
$$z = 1025r^2 + s^2 + 50rs$$

Which represent the integer solutions to (1)

Way 3

`Write (11) as the system of double equations as shown in Table: 1 below

Table 1: System of Double Equations

System	1	2	3	4	5	6	7	8	9
$X+U$	$1025T^2$	$205T^2$	$41T^2$	$25T^2$	$5T^2$	T^2	$1025T$	$205T$	$41T$
$X-U$	1	5	25	41	205	1025	T	$5T$	$25T$

Solving each of the system of double equations in Table: 1, the values of X, T and U are obtained. In view of (10) and (2), the corresponding integer solutions to (1) are obtained. For simplicity and brevity, the corresponding integer solutions to (1) obtained from each of the system of double equations from Table: 1 are presented.

Solutions for System 1:

$$x = 10250k^2 + 10332k + 2602$$
$$y = 6150k^2 + 6068k + 1494$$
$$z = 2050k^2 + 2100k + 538$$

Solutions for System 2:

$$x = 2050k^2 + 2132k + 546$$
$$y = 1230k^2 + 1148k + 254$$
$$z = 410k^2 + 460k + 130$$

Solutions for System 3:

$$x = 410k^2 + 492k + 26$$
$$y = 246k^2 + 164k - 122$$
$$z = 82k^2 + 132k + 58$$

Solutions for System 4:

$$x = 250k^2 + 332k + 42$$
$$y = 150k^2 + 68k - 106$$
$$z = 50k^2 + 100k + 58$$

Solutions for System 5:

$$x = 50k^2 + 132k - 254$$
$$y = 30k^2 - 52k - 546$$
$$z = 10k^2 + 60k + 130$$

Solutions for System 6:

$$x = 10k^2 + 92k - 1494$$
$$y = 6k^2 - 76k - 2602$$
$$z = 2k^2 + 52k + 538$$

Solution for System 7:

$$x = 2602k, y = 1494k, z = 538k$$

Solution for System 8:

$$x = 546, y = 254k, z = 130k$$

Solution for System 9:

$$x = 106k, y = -42k, z = 58k$$

Way 4:

Taking

$$u = 5U, z = 5w \tag{13}$$

in (3), it gives

$$U^2 + v^2 = 41w^2 \tag{14}$$

Let

$$w = a^2 + b^2 \tag{15}$$

Write 41 as

$$41 = (5 + i4)(5 - i4) \tag{16}$$

Substituting (15) and (16) in (14) and employing the method of factorization, define

$$U + iv = (5 + i4)(a + ib)^2$$

From which one obtains

$$\left.\begin{aligned} U &= 5(a^2 - b^2) - 8ab \\ v &= 4(a^2 - b^2) + 10ab \end{aligned}\right\} \tag{17}$$

From (17), (15), (13) and (2), observe that (1) is satisfied by

$$x = 29(a^2 - b^2) - 30ab$$
$$y = 21(a^2 - b^2) - 50ab$$
$$z = 5(a^2 + b^2)$$

Note: 1

One may write 41 on the R.H.S. of (14) as

$$41 = (4 + i5)(4 - i5)$$

Following the procedure presented above in WAY4, the corresponding integer solutions to (1) are found to be

$$x = 25(a^2 - b^2) - 42ab$$
$$y = 15(a^2 - b^2) - 58ab$$
$$z = 5(a^2 + b^2)$$

Way 5:

Rewrite (14) as

$$41w^2 - v^2 = U^2 = U^2 * 1 \tag{18}$$

Let

$$U = 41a^2 - b^2 \tag{19}$$

Write 1 as

$$1 = \frac{(\sqrt{41}+5)(\sqrt{41}-5)}{16} \tag{20}$$

Substituting (19) and (20) in (18) and employing the method of factorization, define

$$\sqrt{41}w + v = \frac{(\sqrt{41}+5)}{4}(\sqrt{41}a + b)^2$$

from which one has

$$\left.\begin{aligned} v &= \frac{1}{4}[5(41a^2 + b^2) + 82ab] \\ w &= \frac{1}{4}[41a^2 + b^2 + 10ab] \end{aligned}\right\} \tag{21}$$

As our interest is on finding integer solutions, replacing a by 2A and b by 2B in (19) and (21), observe that

$$\left.\begin{aligned} U &= 4(41A^2 - B^2) \\ v &= 5(41A^2 + B^2) + 82AB \\ w &= 41A^2 + B^2 + 10AB \end{aligned}\right\} \tag{22}$$

From (22), (13) and (2), it is seen that (1) is satisfied by

$$x = 1025A^2 - 15B^2 + 82AB$$
$$y = 615A^2 - 25B^2 - 82AB$$
$$z = 5(41A^2 + B^2 + 10AB)$$

Note: 2

One may write 1 on the R.H.S. of (18) as

$$1 = \frac{(\sqrt{41}+4)(\sqrt{41}-4)}{25}$$

For this choice, the corresponding integer solutions to (1) are given by

$$x = 5945A^2 - 105B^2 + 410AB$$
$$Y = 4305A^2 - 145B^2 - 410AB$$
$$z = 25(41A^2 + B^2 + 8AB)$$

Way 6:

Rewrite (14) in the form of ratio as

$$\frac{U+4w}{5w+v} = \frac{5w-v}{U-4w} = \frac{\sim}{\quad}, \quad \neq 0$$

which is equivalent to the system of double equations

$$U - \sim v + (4 \quad - 5\sim)w = 0$$
$$\sim U + \quad v - (4\sim + 5 \quad)w = 0$$

Solving the above system by the method of cross-multiplication, it is seen that

$$\left.\begin{aligned} U &= 4\sim^2 + 10\sim \quad - 4 \quad{}^2 \\ V &= -5\sim^2 + 8\sim \quad + 5 \quad{}^2 \\ w &= \quad{}^2 + \sim^2 \end{aligned}\right\} \tag{23}$$

From (23), (13) and (2), integer solutions to (1) are found to be

$$x = 15\sim^2 + 50\sim \quad - 15 \quad{}^2$$
$$y = 25\sim^2 + 42\sim \quad - 25 \quad{}^2$$
$$z = 5(\quad{}^2 + \sim^2)$$

Note: 3

One may write (14) in the form of ratio as

$$\frac{U+5w}{4w+v} = \frac{4w-v}{U-5w} = \frac{\sim}{\quad}, \quad \neq 0$$

In this case, the corresponding integer solutions to (1) are given by

$$x = 21\sim^2 + 50\sim \quad - 21 \quad{}^2$$
$$y = 29\sim^2 + 30\sim \quad - 29 \quad{}^2$$
$$z = 5(\quad{}^2 + \sim^2)$$

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